Poincaré inequalities revisited for dimension reduction

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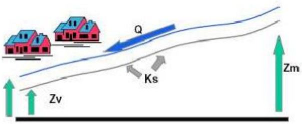
Part I

Background and motivation

A case study for global sensitivity analysis



A simplified flood model [Iooss, 2011], [Iooss and Lemaitre, 2015].

- 1 output: maximal annual overflow (in meters), denoted by S:

$$S = Z_v + H - H_d - C_b \quad \text{with} \quad H = \left(\frac{Q}{BK_s \sqrt{\frac{Z_m - Z_v}{L}}} \right)^{0.6}$$

where H is the maximal annual height of the river (in meters).

A case study for global sensitivity analysis

- 8 inputs variables assumed to be independent r.v., with distributions:

Input	Description	Unit	Probability distribution
$X_1 = Q$	Maximal annual flowrate	m^3/s	Gumbel $\mathcal{G}(1013, 558)$, truncated on $[500, 3000]$
$X_2 = K_s$	Strickler coefficient	-	Normal $\mathcal{N}(30, 8^2)$, truncated on $[15, +\infty[$
$X_3 = Z_v$	River downstream level	m	Triangular $\mathcal{T}(49, 50, 51)$
$X_4 = Z_m$	River upstream level	m	Triangular $\mathcal{T}(54, 55, 56)$
$X_5 = H_d$	Dyke height	m	Uniform $\mathcal{U}[7, 9]$
$X_6 = C_b$	Bank level	m	Triangular $\mathcal{T}(55, 55.5, 56)$
$X_7 = L$	River stretch	m	Triangular $\mathcal{T}(4990, 5000, 5010)$
$X_8 = B$	River width	m	Triangular $\mathcal{T}(295, 300, 305)$

- *Aim: To detect unessential X_i's, to quantify the influence of X_i's on S, ...*

A case study for global sensitivity analysis

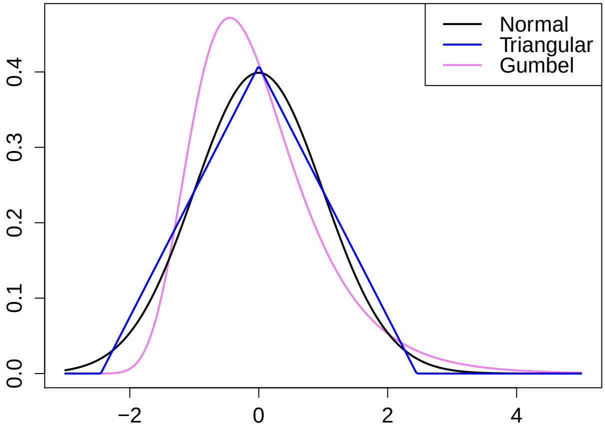


Figure: The 3 distributions types of the case study, here with mean 0 and variance 1

Towards variance-based sensitivity measures

Framework. $X = (X_1, \dots, X_d)$ is a vector of independent input variables with distribution $\mu_1 \otimes \dots \otimes \mu_d$, and $g : \Delta \subseteq \mathbb{R}^d \to \mathbb{R}$ is such that $g(\mathbf{X}) \in L^2(\mu)$.

Sobol-Hoeffding decomposition [Sobol, 1993, Efron and Stein, 1981]

$$g(\mathbf{X}) = g_0 + \sum_{i=1}^{d} g_i(X_i) + \sum_{1 \le i < j \le d} g_{i,j}(X_i, X_j) + \dots + g_{1,\dots,d}(X_1, \dots, X_d)$$

The g_I's satisfy $E[g_I(X_I)|X_J] = 0$ if $J \subsetneq I$, implying orthogonality.
They are obtained sequentially via

$$\mathbb{E}(g(\mathbf{X})|\mathbf{X}_I) = \int_{\mathbb{R}^{d-|I|}} g(\mathbf{x}) d\mu_{-I}(\mathbf{x}_{-I})$$

Variance-based and derivative-based measures

Variance decomposition and Sobol indices

- Partial variances: $D_I = \mathrm{Var}(g_I(X_I))$, and **Sobol indices** $S_I = D_I / D$

$$D := \mathrm{Var}(g(\mathbf{X})) = \sum_I D_I, \qquad 1 = \sum_I S_I$$

- **Total index**: $D_i^{\mathrm{T}} = \sum_{J \supseteq \{i\}} D_J, \qquad S_i^{\mathrm{T}} = \frac{D_i^{\mathrm{T}}}{D}.$

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Derivative Global Sensitivity Measure (DGSM), [Sobol and Gershman, 1995], [Kucherenko et al., 2009]

$$\nu_i = \int \left(\frac{\partial g(\mathbf{x})}{\partial x_i} \right)^2 d\mu(\mathbf{x})$$

Variance-based and derivative-based measures

- **Usage for screening.**
  If either $D_i^T = 0$ or $\nu_i = 0$, than X_i is non influential

- **Advantages / Drawbacks**

	Computational cost	Interpretability
Sobol indices	-	+
DGSM	+	-

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$\downarrow$

Can we use DGSM to do screening based on Sobol indices?

Poincaré inequality

Poincaré inequality (1-dimensional case)

A distribution μ satisfies a Poincaré inequality if the energy in $L^2(\mu)$ sense of any centered function is controlled by the energy of its derivative:

For all h in $L^2(\mu)$ such that $\int h(x)d\mu(x) = 0$, and $h'(x) \in L^2(\mu)$:

$$\int h(x)^2 d\mu(x) \leq C(\mu) \int h'(x)^2 d\mu(x)$$

The best constant is denoted $C_{\mathrm{P}}(\mu)$.

Link between total Sobol indices and DGSM

Theorem [Lamboni et al., 2013], extended in [Roustant et al., 2014]

If μ_i admits a Poincaré inequality, then there is a Poincaré-type inequality between total indices and DGSMs

$$D_i \;\leq\; D_i^{\mathrm{T}} \leq\; C(\mu_i)\nu_i$$

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Proof. Denote $g_i^{\mathrm{T}}(\mathbf{x}) := \sum_{J \supseteq \{i\}} g_J(\mathbf{x}_J)$. Then:

$$\frac{\partial g(\mathbf{x})}{\partial x_i} = \frac{\partial g_i^{\mathrm{T}}(\mathbf{x})}{\partial x_i}$$

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$$\frac{\partial g(\mathbf{x})}{\partial x_i} = \frac{\partial g_i^{\mathrm{T}}(\mathbf{x})}{\partial x_i}$$

$$\begin{aligned} D_i^T = \mathrm{Var}(g_i^{\mathrm{T}}(\mathbf{x})) \quad &= \quad \int \left(g_i^{\mathrm{T}}(\mathbf{x})\right)^2 d\mu(\mathbf{x}) \\ &\leq \quad C(\mu_i) \int \left(\frac{\partial g^{\mathrm{T}}(\mathbf{x})}{\partial x_i}\right)^2 d\mu(\mathbf{x}) = C(\mu_i)\nu_i \end{aligned}$$

'Low-cost' screening based on Sobol indices via DGSM

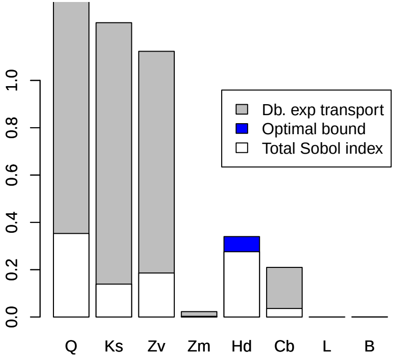

Figure: Total Sobol index S_i^T & Upper bound $C(\mu_i)\nu_i/D$

'Low-cost' screening based on Sobol indices via DGSM

- **Even if the Poincaré inequality is not accurate,**
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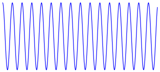
- **When is the upper bound $C(\mu_i)\nu_i$ large (compared to D_i^T)?**

ν_i is large

$\downarrow$

Ex: high frequency in g w.r.t. x_i

...we cannot do anything!



$C(\mu_i)$ is large

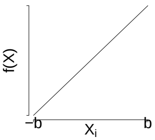
$\downarrow$

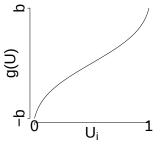
We can look for smallest $C(\mu_i)$

Why not transforming the problem to get uniform distributions?

Example with X_1, X_2 i.i.d. $\mathcal{N}(0,1)$ truncated on $I = [-b, b]$

$$f(X_1, X_2) = X_1 + X_2 \quad \Big| \quad g(U_1, U_2) = F_X^{-1}(U_1) + F_X^{-1}(U_2)$$





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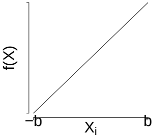

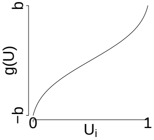

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Difference on *optimal* upper bounds computed with DGSM

$\mu(I)$	S^T	Upper bound with DGSM Original problem (with f)	Upper bound with DGSM Transformed problem (with g)
1	0.5	0.5	$+\infty$
0.95	0.5	0.52	1.48
0.75	0.5	0.56	1.00

The derivatives are larger on the transformed problem $\Rightarrow$ larger bounds

1D Poincaré constants: Known results

pdf	Support	$C_{\mathrm{P}}(\mu)$	$f_{\mathrm{opt}}(x)$
Uniform	$[a, b]$	$(b-a)^2/\pi^2$	$\cos\left(\frac{\pi(x-a)}{b-a}\right)$
$\mathcal{N}(m, s^2)$	$\mathbb{R}$	s^2	$x - m$
	$[r_{n,i}, r_{n,i+1}]$ (*)	$1/(n+1)$	$H_{n+1}(x)$
Double exp. $e^{-\lvert x\rvert}dx/2$	$\mathbb{R}$	4	×
Logistic $\frac{e^x}{(1+e^x)^2}dx$	$\mathbb{R}$	4	×

(*) H_n is the Hermite polynomial of degree n, and $r_{n,1}, \ldots, r_{n,n}$ its zeros.

Summary and aim

1. *Sobol indices and DGSM are linked by Poincaré-type inequalities*

$$D_i^T \leq C(\mu_i)\nu_i \qquad D_{i,j}^{\text{super}} \leq C(\mu_i)C(\mu_j)\nu_{i,j}$$

2. DGSM are easier to compute but Sobol indices are more interpretable
$\Rightarrow$ *DGSM may allow doing low-cost screening based on Sobol indices*

3. The aim: *To look for the exact Poincaré constants for distributions met in practice*:
   - Frequently: Uniform – (truncated) Gaussian – Triangular – (truncated) lognormal – Exponential – (truncated) Weibull – (truncated) Gumbel
   - Less frequently: (Inverse) Gamma – Beta – Trapezoidal – Generalized Extreme Value

Outline

Part II

Theory for optimal inequalities

Mathematical setting

Definitions and assumptions

- Ω: an open interval (a, b) of $\mathbb{R}$ (possibly unbounded)
- $\mu(dt) = \rho(t)dt$: A continuous measure supported by Ω.
  $\rho > 0$ on Ω, continuous on $\overline{\Omega}$ and piecewise C^1 on Ω.
- *f': weak derivative of f*, i.e. s.t. for all ϕ of class C^∞ with compact support

$$\int_\Omega f(t)\phi'(t)dt = -\int_\Omega f'(t)\phi(t)dt.$$

- *Sobolev spaces*
  - $\mathcal{H}^1_\mu(\Omega) = \{f \in L^2(\mu) \text{ such that } f' \in L^2(\mu)\}$
  - $\mathcal{H}^\ell_\mu(\Omega) = \{f \in L^2(\mu) \text{ such that for all } k \leq \ell, f^{(k)} \in L^2(\mu)\}$

Mathematical setting

When weighted Sobolev spaces collapse to usual Sobolev spaces

Assume that μ is a bounded perturbation of $\mathcal{U}(\Omega)$, i.e. $0 < m < \rho(t) < M$.
Then:

$$L^2(\mu) = L^2(\mathcal{U}(\Omega)) \qquad \mathcal{H}^\ell_\mu(\Omega) = \mathcal{H}^\ell_{\mathcal{U}(\Omega)}(\Omega)$$

with equivalent norms.

Poincaré inequality and Rayleigh ratio

Rayleigh ratio

For $f \in \mathcal{H}^1_\mu(\Omega)$:

$$J(f) = \frac{\int_\Omega f'^2 d\mu}{\int_\Omega f^2 d\mu} = \frac{\|f'\|^2}{\|f\|^2}$$

Finding the Poincaré constant is equivalent to:

$$\min_{f \in \mathcal{H}^1_\mu(\Omega)} J(f) \quad s.t. \quad \int_\Omega f \, d\mu = 0$$

and $C_{\mathrm{P}}(\mu)$ denotes the *inverse* of the minimum.

Optimizing a Rayleigh ratio in finite dimensions

Exercice!

Let A a positive definite matrix on $\mathbb{R}^n$. Find:

$$\min_{x \in \mathbb{R}^n} \frac{\|x\|_A^2}{\|x\|^2}$$

with $\|x\|_A^2 = x^T A x$

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Exercice!

Let A a positive definite matrix on $\mathbb{R}^n$. Find:

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Solution. A is diagonalisable in an orthonormal basis u_k:

$$A u_k = \lambda_k u_k$$

with $\lambda_1 \geq \cdots \geq \lambda_n > 0$.
Expand x in the basis: $x = \sum x_k u_k$, then $Ax = \sum \lambda_k x_k u_k$:

$$\frac{\|x\|_A^2}{\|x\|^2} = \frac{\sum \lambda_k x_k^2}{\sum x_k^2} \geq \lambda_1$$

with equality if $x = u_1$.

Spectral interpretation and link to a 2nd-order differential equation

Theorem (Synthesis from [Bobkov and Götze, 2009] and [Dautray and Lions, 1990])

Assume that $\Omega = (a, b)$ is bounded, and that $\rho(t) = e^{-V(t)} > 0$ on $\overline{\Omega} = [a, b]$.
A minimizer f of the Rayleigh ratio is obtained by solving

$$Lf := f'' - V'f' = -\lambda f \qquad \textit{with} \quad f'(a) = f'(b) = 0$$

when $\lambda =: \lambda(\mu)$ *is the smallest possible value, called spectral gap.*
Furthermore, $\lambda(\mu)$ *is a simple eigenvalue and f is strictly monotone*.

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Ideas for the proof. Show the connections between the three problems

P1 Find $f \in \mathcal{H}^1_\mu(\Omega)$ s.t. $\quad J(f) = \frac{\|f'\|^2}{\|f\|^2}$ is minimum under $\int f d\mu = 0$

P2 Find $f \in \mathcal{H}^1_\mu(\Omega)$ s.t. $\quad \langle f', g' \rangle = \lambda \langle f, g \rangle \quad \forall g \in \mathcal{H}^1_\mu(\Omega)$

P3 Find $f \in \mathcal{H}^2_\mu(\Omega)$ s.t. $\quad f'' - V'f' = -\lambda f$ and $f'(a) = f'(b) = 0$

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$(P1) \iff (P2)$ (for the smallest positive λ). Start from (P2):

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Up to a switch in order to make coercive the left side, we can 'diagonalize': there exists a Hilbert basis $(u_k)_{k \geq 0}$ and an increasing sequence $(\lambda_k)_{k \geq 0}$ of positive numbers that tends to infinity such that:

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Remark that:

- $\lambda_k = 0 \iff k = 0$, and $u_0 = 1$
- $\int f d\mu = 0 \iff \langle f, 1 \rangle = 0$

Thus f is written: $f = \sum_{k \geq 1} f_k u_k$.

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Thus f is written: $f = \sum_{k\geq 1} f_k u_k$. Finally,

$$J(f) = \frac{\|f'\|^2}{\|f\|^2} = \frac{\sum_{k=1}^{+\infty} \lambda_k f_k^2}{\sum_{k=1}^{+\infty} f_k^2} \geq \lambda_1 > 0$$

with equality iff $f \in \mathbb{R}\, u_1$.

Spectral interpretation and link to a 2nd-order differential equation

$(P2) \iff (P3)$ Formally the link comes from an integration by part (IPP):

$$\langle f', g' \rangle = \int_a^b f' g' \rho = [(f'\rho)g]_a^b - \int_a^b (f'\rho)' g$$

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Thus the two are equal for all $g \in \mathcal{H}^1_\mu(\Omega)$ iff:

- $f'(a) = f'(b) = 0$ (since $\rho > 0$)
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This proves $(P2) \Leftarrow (P3)$, since f is regular enough (IPP is valid).
Conversely, an argument of regularity is necessary. One can show that if f is solution of $(P2)$, then $(f'\rho)$ is of class C^1, and more precisely:

$$f'(x) = \frac{\lambda}{\rho(x)} \int_x^b f(t)\rho(t)dt.$$

Spectral interpretation and link to a 2nd-order differential equation

Neumann and Dirichlet spectral gaps

If f is enough derivable, finding $f \uparrow$ of the Neumann spectral problem

$$Lf := f'' - V'f' = -\lambda f, \qquad f'(a) = f'(b) = 0$$

is equivalent to finding $g > 0$ of the *Dirichlet* spectral problem

$$Kg := g'' - V'g' - V''g = -\lambda g, \qquad g(a) = g(b) = 0$$

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Main idea. Consider $g = f'$

$$\begin{array}{ccc} (Lf)' = f''' - V'f'' - V''f' = -\lambda f' & \qquad & f'(a) = f'(b) = 0 \\ \Updownarrow & & \Updownarrow \\ Kg = g'' - V'g' - V''g = -\lambda g & & g(a) = g(b) = 0 \end{array}$$

Other useful properties: Optimal constants on intervals

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Consequence.
We can assume that $\Omega = (a, b)$ is bounded and that ρ does not vanish on $\overline{\Omega}$.

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Sketch of proof for monotonicity.
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$$\begin{aligned} \mathrm{Var}_{\mu_{|I}}(f) &= \inf_a \int_I (f-a)^2 \frac{d\mu}{\mu(I)} \\ &\leq \inf_a \int_J (\tilde{f}-a)^2 \frac{d\mu}{\mu(I)} \end{aligned}$$

Other useful properties: Optimal constants on intervals

Sketch of proof for monotonicity.
For $f \in \mathcal{H}^1(\mu_{|I})$, extend it on J with a constant outside I. This is $\tilde{f}$.

$$\begin{aligned} \mathrm{Var}_{\mu_{|I}}(f) &= \inf_a \int_I (f-a)^2 \frac{d\mu}{\mu(I)} \\ &\leq \inf_a \int_J (\tilde{f}-a)^2 \frac{d\mu}{\mu(I)} \leq C_{\mathrm{P}}(\mu) \int_J (\tilde{f}')^2 \frac{d\mu}{\mu(I)} \end{aligned}$$

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Sketch of proof for continuity.

- From monotonicity, $C_{\mathrm{P}}(\mu_{|I_\epsilon}) \leq C_{\mathrm{P}}(\mu_{|I})$
- Then, choose $f \in \mathcal{H}^1(\mu_{|I})$, and check with Lebesgue theorem that:

$$\frac{\mathrm{Var}_{\mu_{|I}}(f)}{\int_\Omega f'^2 d\mu_{|I}} = \lim_{\epsilon \to 0} \frac{\mathrm{Var}_{\mu_{|I_\epsilon}}(f)}{\int_{I_\epsilon} f'^2 d\mu_{|I_\epsilon}} \leq \lim_{\epsilon \to 0} C_{\mathrm{P}}(\mu_{|I_\epsilon})$$

Properties for symmetric measures on symmetric intervals

Let $I = (-a, a)$ be a symmetric interval, and μ an even measure on $\mathbb{R}$.

1. *Improved result for monotonicity*

$$C_{\mathrm{P}}(\mu_{|I}) \leq \mu(I)^2 C_{\mathrm{P}}(\mu)$$

Properties for symmetric measures on symmetric intervals

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1. *Improved result for monotonicity*

$$C_{\mathrm{P}}(\mu_{|I}) \leq \mu(I)^2 C_{\mathrm{P}}(\mu)$$

2. *Symmetry and odd functions*

   The infemum of the Rayleigh ratio on I can be found among odd functions

Part III

Semi-analytical results

An example: The truncated normal distribution

Poincaré constant of the truncated Normal distribution

Define:

$$h_0(\lambda, t) = M_{\frac{1-\lambda}{2};\frac{1}{2}}\left(\frac{t^2}{2}\right) \qquad h_1(\lambda, t) = M_{\frac{2-\lambda}{2};\frac{3}{2}}\left(\frac{t^2}{2}\right)$$

where the so-called *Kummer's function* $M_{a_1,b_1}(z) = {}_1F_1\left(a_1; b_1; z\right)$ is an example of *hypergeometric series* $\sum_{p\geq 0} x_p$ satisfying

$$\frac{x_{p+1}}{x_p} = \frac{(p+a_1)z}{(p+b_1)(p+1)}, \qquad x_0 = 1$$

Notice that *h_0 and h_1 generalize Hermite polynomials*: When λ is an odd (resp. even) positive integer, then $t \mapsto h_0(\lambda, t)$ (resp. $t \mapsto t.h_1(\lambda, t)$) is proportional to the Hermite polynomial of degree $\lambda - 1$.
Then the spectral gap of $\mathcal{N}(0,1)_{|[a,b]}$ is the first zero of the function

$$d(.) = bh_0(., a)h_1(., b) - ah_0(., b)h_1(., a)$$

Sketch of proof. Consider the Dirichlet problem

$$g'' - t\,g' = -(\lambda - 1)g, \qquad g(a) = g(b) = 0$$

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  Split g into even an odd part, $g(t) = g_0(t) + t.g_1(t)$. Then:

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- Now, $g(a) = g(b) = 0$ lead to the linear system $Xc = 0$, with:

$$X = \begin{pmatrix} h_0(\lambda, a) & ah_1(\lambda, a) \\ h_0(\lambda, b) & bh_1(\lambda, b) \end{pmatrix} \qquad c = \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}$$

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  This system must be singular, otherwise g would be identically zero.
  Thus $\det(X) = 0$, leading to $d(\lambda) = 0$.

Truncated normal distribution – Symmetric case: I = [-b,b]

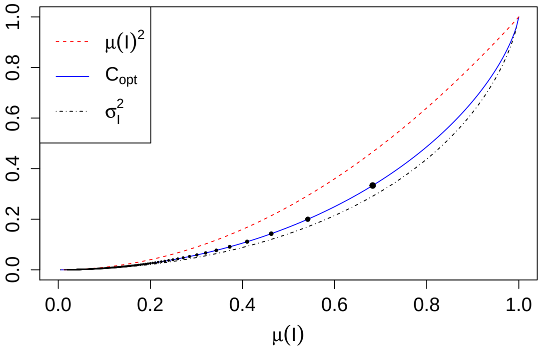

Figure: Poincaré constant of $\mu = \mathcal{N}(0,1)$ truncated on $I = [-b, b]$, vs $\mu(I)$

σ_I^2 : variance of the truncated normal on I – Black points: Hermite polynomials of even degree.

Truncated normal distribution – General case

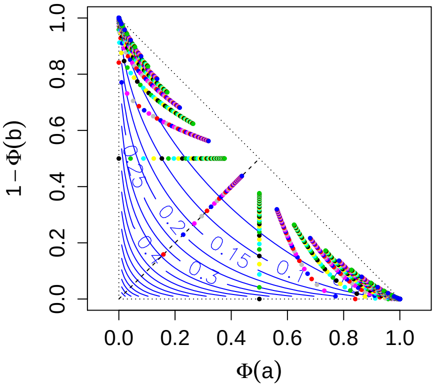

Figure: Poincaré constant of $\mathcal{N}(0,1)$ truncated on $I = [a, b]$.

Colored points: Hermite polynomials (up to degree 100).

Summary: General methodology for finding optimal constants on $[a, b]$

1. Consider the spectral problem $f'' - V'f' = -\lambda f \qquad f'(a) = f'(b) = 0$
2. *Find a basis of 2 independent solutions* $f_{1,\lambda}(t), f_{2,\lambda}(t)$
3. The Neumann conditions lead to a *singular* linear system

$$\begin{pmatrix} f'_{1,\lambda}(a) & f'_{2,\lambda}(a) \\ f'_{1,\lambda}(b) & f'_{2,\lambda}(b) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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Variants.

- Dirichlet problem (ex: Truncated Gaussian)
- Symmetry (ex: Triangular)

1D Poincaré constants: Additional results

pdf	Support	C_{opt}	Form of $f_{\text{opt}}(x)$
Uniform	$[a, b]$	$(b-a)^2/\pi^2$	$\cos\left(\frac{\pi(x-a)}{b-a}\right)$
$\mathcal{N}(\mu, \sigma^2)$	$\mathbb{R}$	σ^2	$x - \mu$
	$[r_{n,i}, r_{n,i+1}]$	$1/(n+1)$	$H_{n+1}(x)$
	$[a, b]$	*see before*	*related to Kummer*
Db. exp. $e^{-\lvert x\rvert}dx/2$	$\mathbb{R}$	4	$\times$
(*)	$[a, b], ab > 0$	$\left(\frac{1}{4} + \omega^2\right)^{-1}$	$e^{x/2}\cos(\omega x + \phi)$
(*, **)	$[a, b], ab \leq 0$	$> \left(\frac{1}{4} + \omega^2\right)^{-1}$	$e^{\lvert x\rvert/2} \times$ *trig. spline*
Logistic $\frac{e^x}{(1+e^x)^2}dx$	$\mathbb{R}$	4	$\times$
Triangular	$[-1, 1]$	≈ 0.1729	*linked to Bessel* J_0

(*) For the truncated Exponential on $[a, b] \subseteq \mathbb{R}^+$, we use $\omega = \pi/(b-a)$
(**) If $a < 0 < b$, the spectral gap is the zero in $]0, \min(\pi/|a|, \pi/|b|)[$ of $x \mapsto \mathrm{cotan}(|a|x) + \mathrm{cotan}(|b|x) + 1/x$

Part IV

A numerical method

Principle: Finite elements

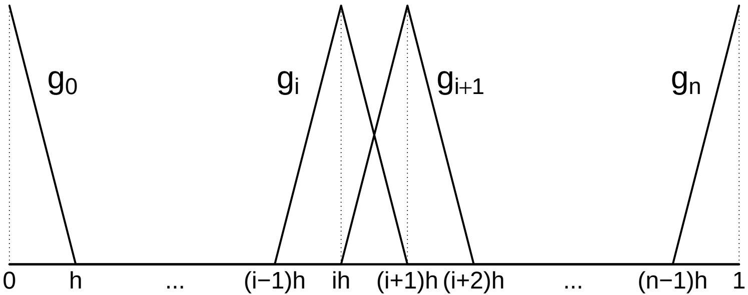

Figure: Basis of finite elements $\mathbb{P}_1$ on $[0, 1]$. The g_i's are hat functions for $i = 1, \ldots, n - 1$, truncated at the boundaries ($i = 0$ and $i = n$).

Principle: Finite elements

The idea is to solve numerically the spectral problem (P2)

$$\langle f', g' \rangle = \lambda \langle f, g \rangle \qquad \forall g \in \mathcal{H}^1_\mu(\Omega)$$

Solutions are obtained as the limit of solutions in the finite dim. space $\mathbb{P}_1$.

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In $\mathbb{P}_1$, the problem is to find $\mathbf{f_h} \in \mathbb{R}^{n+1}$ such that

$$K_h \mathbf{f_h} = \lambda M_h \mathbf{f_h}$$

with: $K_h = (\langle g_i', g_j' \rangle)_{0 \leq i,j \leq n}$ and $M_h = (\langle g_i, g_j \rangle)_{0 \leq i,j \leq n}$.

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Using the Choleski dec. of $M_h = L_h L_h^T$, we obtain an eigenvalues problem

$$\widetilde{K_h} \widetilde{\mathbf{f_h}} = \lambda \widetilde{\mathbf{f_h}}$$

with $\widetilde{K_h} = L_h^{-1} K_h (L_h^T)^{-1}$ and $\widetilde{\mathbf{f_h}} = L_h^T \mathbf{f_h}$.

Convergence properties

Proposition

Assume that Ω is bounded and $\rho > 0$ on $\overline{\Omega}$.
Consider the solutions of the spectral problem in $\mathbb{P}_1$,

$$0 = \lambda_{0,h} \leq \lambda_{1,h} \leq \cdots \leq \lambda_{n,h}$$

and $u_{0,h}, u_{1,h}, \ldots, u_{n,h}$. corr. eigenvectors. Let $\ell \geq 1$ s.t. $f_{\text{opt}} \in \mathcal{H}_\mu^{\ell+1}(\Omega)$. Then:

$$|\lambda_{1,h} - \lambda(\mu)| = O(h^{2\ell}), \qquad |u_{1,h} - f_{\text{opt}}| = O(h^{\ell})$$

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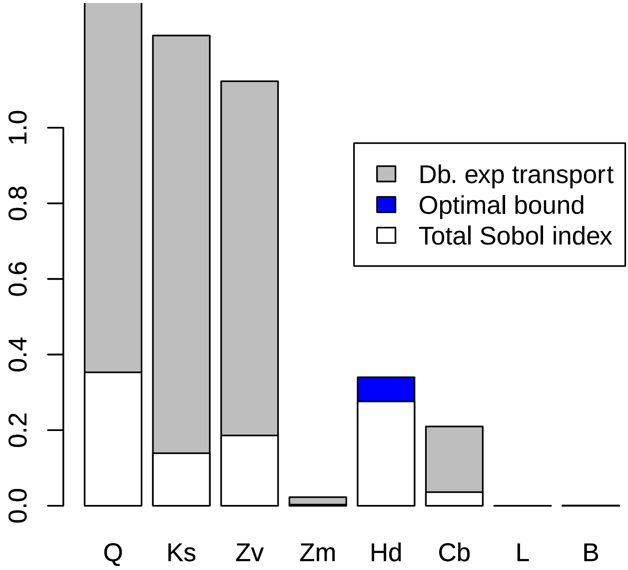
This also applies under the assumptions above since μ is a bounded perturbation of $\mathcal{U}(\Omega)$: $0 < m < \rho(t) < M$.

Part V

Applications

Come-back to the case study



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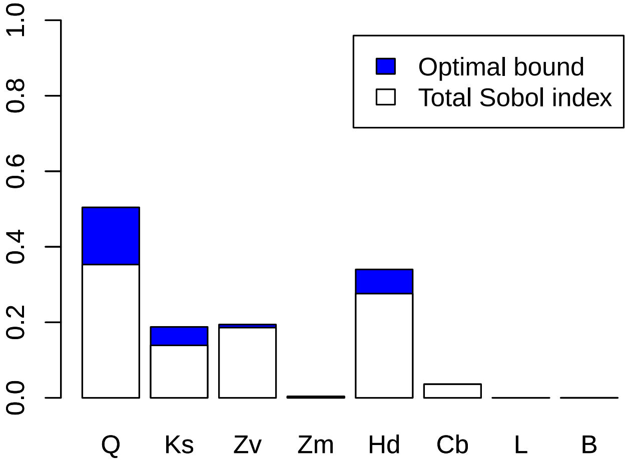
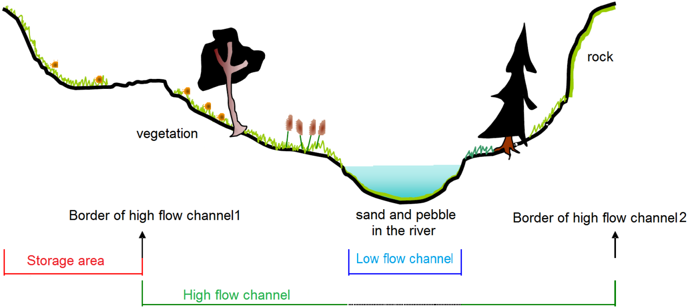


Application on a 1D hydraulic model



- Mascaret simulator on Vienne river (Saint Venant Lab.)
- $d = 37$ random inputs (uniform and truncated Gaussian)
- Output: The water level at a specific river location
- Adjoint model gives derivatives (cost independent of d) and DGSM [Petit et al., 2016]

Application on a 1D hydraulic model

Study with $n = 20,000$ on 5 inputs previously identified as active

Inputs	$K_{s,c}^{11}$	$K_{s,c}^{12}$	dZ^{11}	dZ^{12}	Q
μ	$\mathcal{U}$	$\mathcal{U}$	$\mathcal{T}N$	$\mathcal{T}N$	$\mathcal{T}N$
S^T	0.456	0.0159	0.293	0.015	0.239
	(2e−3)	(1e−4)	(1e−3)	(1e−4)	(1e−3)
	By double exponential transport				
Upper bound	-	-	1.844	0.116	1.504
	-	-	(2e−3)	(2e−3)	(1.5e−2)
	By logistic transport				
Upper bound	-	-	0.461	0.028	0.376
	-	-	(4e−3)	(5e−4)	(4e−3)
	Optimal Poincaré constant				
Optimal bound	0.625	0.029	0.288	0.017	0.235
	(2e−4)	(1e−5)	(3e−3)	(3e−4)	(2e−3)

Part VI

Conclusion

Conclusions

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   $\Rightarrow$ Will work if the function is not varying too quickly

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1. *DGSM allow doing low-cost screening based on Sobol indices*
   $\Rightarrow$ Will work if the function is not varying too quickly

2. *$C_{\mathrm{P}}(\mu)$ can be computed semi-analytically for simple distributions*, e.g. *in blue* in our initial list:
   - Frequently: *Uniform* – *(truncated) Gaussian* – *Triangular* – (truncated) lognormal – *truncated Exp.* – (truncated) Weibull – (truncated) Gumbel
   - Less frequently: (Inverse) Gamma – Beta – Trapezoidal – Generalized Extreme Value

3. *$C_{\mathrm{P}}(\mu)$ can be computed numerically with finite elements*.

See more details on our preprint
`https://hal.archives-ouvertes.fr/hal-01388758`

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We are also grateful to:

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- Laurence Grammont and the members of the team 'Génie Mathématique & Industriel' for useful discussions.
- The participants of the previous conferences where this research was presented!

Part VII

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